EVALUATION OF UNIVERSAL FUNCTIONS FOR THE RADIANT FLUX TRANSMISSION

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UDC 536.33

Universal functions of radiant flux transmission are evaluated for a wide range of optical thicknesses. An approximate engineering formula is derived for calculating the flux in a plane layer of gray substance.

The equation of the temperature field in a plane gray nondispersive layer with a constant heat source and with an absorptivity which is not temperature-dependent can be transformed, if the unknown quantities are two functions [1, 2], into

$$\varphi(\tau,\ \tau_0)=\frac{1}{2}E_2(\tau)+\frac{1}{2}\int_0^{\tau_0}\varphi(t,\ \tau_0)\,E_1(|\tau-t|)\,dt, \tag{1}$$

$$\varphi_s(\tau,\ \tau_0)=\frac{1}{4}+\frac{1}{2}\int_0^{\tau_0}\varphi_s(t,\ \tau_0)\,E_1(|\tau-t|)\,dt. \tag{2}$$

Analogously transformed equations describing the temperature field in a layer of selective material with an absorptivity of the Milne–Eddington kind has been shown in [3].

The thermal flux through a layer of gray material can be easily determined with the aid of the known functions

$$Q(\tau_0)=1-2\int_0^{\tau_0}\varphi(t,\ \tau_0)\,E_2(t)\,dt, \tag{3}$$

$$Q_s(\tau_0)=2\int_0^{\tau_0}\varphi_s(t,\ \tau_0)\,E_2(t)\,dt=\frac{\tau_0}{2}, \tag{4}$$

and analogously for selective materials

$$Q'(\tau_0)=2E_3(\tau)+2\int_0^{\tau_0}\varphi(t,\ \tau_0)\,\mathrm{sign}\,(\tau-t)\,E_2(|\tau-t|)\,dt, \tag{5}$$

$$Q_s'(\tau,\ \tau_0)=2\int_0^{\tau_0}\varphi_s(t,\ \tau_0)\,\mathrm{sign}\,(\tau-t)\,E_2(|\tau-t|)\,dt. \tag{6}$$

We note that $Q(\tau_0) = Q'(\tau_0)$. The values of $Q(\tau_0)$ have been tabulated in [2]. The solution which we have obtained for $Q_S'(\tau,\ \tau_0)$ is

$$Q_s'(\tau,\ \tau_0)=\tau-\tau_0/2. \tag{7}$$

The integral equations (1) and (2) have been solved in [2] using the Hopf function and moments of the Chandrasekary–Ambartsumyan function. Despite the high accuracy of these solutions, however, they are difficult to use in practical calculations, because the functions which the solutions $\varphi(\tau,\ \tau_0)$ and $\varphi_S(\tau,\ \tau_0)$ contain are given in tabulated form. It suffices to note that $\varphi(\tau,\ \tau_0)$ contains three such tabulated functions and $\varphi_S(\tau,\ \tau_0)$ contains six, moreover one of the functions in the solution $\varphi_S(\tau,\ \tau_0)$ tends to infinity when

Evening Institute of Metallurgy, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 23, No. 1, pp. 26-32, July, 1972. Original article submitted September 29, 1971.

TABLE 1. Universal $\varphi(\tau, \tau_0)$ Functions

τ/τ_0	τ_0												
	0,01	0,02	0,04	0,06	0,08	0,10	0,20	0,30	0,40	0,50	0,60	0,80	1,00
0	0,5126	0,5217	0,5366	0,5490	0,5598	0,5694	0,6114	0,6420	0,6666	0,6873	0,7052	0,7346	0,7581
0,05	0,5111	0,5190	0,5319	0,5426	0,5518	0 5599	0,5967	0,6229	0,6441	0,6619	0,6773	0,7026	0,7230
0,10	0,5097	0,5167	0,5279	0,5372	0,5451	0,5521	0,5845	0,6073	0,6257	0,6411	0,6546	0,6767	0,6946
0,15	0,5084	0,5145	0,5242	0,5321	0,5389	0,5449	0,5730	0,5927	0,6085	0,6219	0,6335	0,6527	0,6682
0,20	0,5072	0,5123	0,5205	0,5272	0,5330	0 5380	0,5620	0,5787	0,5921	0,6035	0,6133	0,6296	0,6429
0,25	0,5060	0,5102	0,5170	0,5225	0,5273	0,5314	0,5513	0,5651	0,5762	0,5856	0,5937	0,6073	0,6183
0,30	0,5048	0,5081	0,5135	0,5179	0,5217	0,5250	0,5408	0,5518	0,5606	0,5681	0,5746	0,5854	0,5942
0,35	0,5036	0,5061	0,5101	0,5134	0,5162	0,5186	0,5305	0,5387	0 5453	0,5509	0,5557	0,5638	0,5704
0,40	0,5024	0,5040	0,5067	0,5089	0,5108	0,5124	0,5203	0,5257	0,5301	0,5338	0,5371	0,5424	0,5468
0,45	0,5012	0,5020	0,5033	0,5044	0,5054	0,5062	0,5101	0,5128	0,5150	0,5169	0,5185	0,5212	0,5234
0,50	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000

τ/τ_0	τ_0												
	1,5	2,0	2,5	3,0	4,0	5,0	6,0	8,0	10,0	20	30	50	100
0	0,8012	0,8307	0,8527	0,8693	0,8935	0,9101	0,9222	0,9387	0,9494	0,9730	0,9816	0,9888	0,9943
0,05	0,7605	0,7866	0,8060	0,8211	0,8432	0,8587	0,8702	0,8861	0,8967	0,9207	0,9298	0,9376	0,9437
0,10	0,7277	0,7509	0,7683	0,7819	0,8020	0,8162	0,8267	0,8415	0,8513	0,8736	0,8819	0,8889	0,8944
0,15	0,6970	0,7174	0,7328	0,7418	0,7627	0,7754	0,7849	0,7981	0,8069	0,8268	0,8342	0,8403	0,8451
0,20	0,6676	0,6851	0,6984	0,7088	0,7244	0,7354	0,7437	0,7552	0,7629	0,7801	0,7864	0,7917	0,7958
0,25	0,6389	0,6535	0,6647	0,6734	0,6866	0,6959	0,7028	0,7126	0,7190	0,7334	0,7387	0,7431	0,7465
0,30	0,6107	0,6224	0,6314	0,6384	0,6490	0,6565	0,6621	0,6700	0,6752	0,6867	0,6910	0,6945	0,6972
0,35	0,5827	0,5916	0,5983	0,6037	0,6116	0,6173	0,6215	0,6275	0,6314	0,6401	0,6432	0,6459	0,6479
0,40	0,5551	0,5610	0,5655	0,5690	0,5744	0,5782	0,5810	0,5850	0,5876	0,5934	0,5955	0,5972	0,5986
0,45	0,5275	0,5305	0,5327	0,5345	0,5372	0,5391	0,5405	0,5425	0,5438	0,5467	0,5477	0,5486	0.5493
0,50	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000

$\tau_0 \to 0$. We note that data on $\varphi(\tau, \tau_0)$ and $\varphi_S(\tau, \tau_0)$ are given in [2] for $\tau_0 = 0.2$, 1.0, and 2.0; $\varphi(\tau, \tau_0)$ graphs have been plotted in [4] for $\tau_0 = 0.1$, 0.5, 1.0, 2.0, 10, and ∞.

In order to avoid integrating with respect to many functions contained in the solutions $\varphi(\tau, \tau_0)$ and $\varphi_S(\tau, \tau_0)$, the authors have here calculated these solutions on a digital computer for the practical range of optical thicknesses τ_0. The results of the computations are listed in Tables 1 and 2 respectively, with a τ_0 interval convenient for linear interpolation of intermediate values. In calculating $\varphi(\tau, \tau_0)$ and $\varphi_S(\tau, \tau_0)$ for $\tau_0 \geq 0.2$ we used the tables in [2]. Intermediate values of the tabulated functions were found by cubic interpolation from four nodal points in the tables. For $\tau_0 < 0.2$ we calculated $\varphi(\tau, \tau_0)$ and $\varphi_S(\tau, \tau_0)$ by the method of averaging of functional corrections [5]. Preliminary calculations have shown that the first iteration by this method yields a satisfactory accuracy for small optical thickness τ_0. With the aid of the analysis in [6], we will approximate $\varphi(\tau, \tau_0)$ and $\varphi_S(\tau, \tau_0)$ by the following expressions

$$\varphi(\tau, \tau_0) \cong \frac{1}{4}\left[2+E_2(\tau)-E_2(\tau_0-\tau)\right], \tag{8}$$

$$\varphi_s(\tau, \tau_0) \cong \frac{1}{4}\left\{1+\frac{\tau_0}{1-2E_3(\tau_0)}\left[2-E_2(\tau)-E_2(\tau_0-\tau)\right]\right\}. \tag{9}$$

The largest error for $\tau_0 = 0.2$ is then 1.26% in $\varphi(\tau, \tau_0)$ and 0.15% in $\varphi_S(\tau, \tau_0)$. As τ_0 decreases, the error in both approaches zero. It is to be noted that using expression (9) for the determination of $\varphi_S(\tau, \tau_0)$, for example, results in a 1.15% error when $\tau_0 = 0.6$ and a 2.97% error when $\tau_0 = 1$.

The exponential integrals $E_2(z)$ and $E_3(z)$ are calculated by polynomials [7]. In Tables 1 and 2 we show the values of $\varphi(\tau, \tau_0)$ and $\varphi_S(\tau, \tau_0)$ only up to $\tau/\tau_0 = 0.5$, inasmuch as $\varphi[\tau, (\tau_0 - \tau)] = 1 - \varphi(\tau, \tau_0)$ and $\varphi_S[\tau, (\tau_0 - \tau)] = \varphi_S(\tau, \tau_0)$.

It must be noted that an approximate analytical solution for $\varphi(\tau, \tau_0)$ and $\varphi_S(\tau, \tau_0)$ has been obtained in [8], but the results are shown only for $\tau_0 = 0.2$, 1.0, and 2.0 for a comparison with the exact solution.

The approximate analytical expressions for $\varphi(\tau, \tau_0)$ and $\varphi_S(\tau, \tau_0)$ are of interest. When $\tau_0 \leq 0.3$, function $\varphi(\tau, \tau_0)$ can be found according to expression (8). The largest error then does not exceed 3%. When $\tau_0 > 0.3$, the rougher approximation

TABLE 2. Universal $\varphi_S(\tau, \tau_0)$

τ/τ_0	τ_0												
	0,05	0,10	0,20	0,30	0,40	0,50	0,60	0,80	1,0	1,5	2,0	2,5	3,0
0	0,2739	0,2914	0,3217	0,3487	0,3749	0,4000	0,4240	0,4710	0,5168	0,6288	0,7387	0,8479	0,9572
0,05	0,2752	0,2943	0,3280	0,3591	0,3897	0,4197	0,4490	0,5079	0,5674	0,7205	0,8816	1,0517	1,2317
0,10	0,2761	0,2962	0,3321	0,3657	0,3992	0,4324	0,4652	0,5321	0,6006	0,7814	0,9776	1,1900	1,4198
0,15	0,2768	0,2976	0,3353	0,3709	0,4067	0,4424	0,4780	0,5511	0,6268	0,8299	1,0546	1,3018	1,5728
0,20	0,2773	0,2987	0,3378	0,3751	0,4127	0,4504	0,4882	0,5664	0,6480	0,8695	1,1178	1,3942	1,6999
0,25	0,2777	0,2996	0,3399	0,3784	0,4175	0,4568	0,4964	0,5787	0,6652	0,9017	1,1695	1,4699	1,8045
0,30	0,2780	0,3002	0,3415	0,3810	0,4213	0,4619	0,5029	0,5885	0,6787	0,9273	1,2107	1,5307	1,8886
0,35	0,2783	0,3008	0,3427	0,3830	0,4241	0,4657	0,5079	0,5959	0,6891	0,9468	1,2423	1,5773	1,9532
0,40	0,2785	0,3011	0,3435	0,3844	0,4261	0,4684	0,5113	0,6011	0,6963	0,9606	1,2647	1,6102	1,9990
0,45	0,2786	0,3013	0,3440	0,3852	0,4273	0,4700	0,5134	0,6042	0,7007	0,9688	1,2779	1,6299	2,0264
0,50	0,2786	0,3014	0,3442	0,3855	0,4277	0,4706	0,5141	0,6052	0,7021	0,9715	1,2824	1,6365	2,0355

τ/τ_0	τ_0												
	3,5	4,0	5,0	6,0	7,0	8,0	9,0	10	15	20	30	50	100
0	1,0654	1,1737	1,3902	1,6067	1,8232	2,0397	2,2562	2,4727	3,5552	4,6378	6,8028	11,133	21,958
0,05	1,4202	1,6183	2,0426	2,5042	3,0028	3,5380	4,1096	4,7175	8,2956	12,766	24,373	58,242	205,18
0,10	1,6654	1,9281	2,5043	3,1481	3,8593	4,6379	5,4836	6,3966	11,969	19,221	38,774	98,109	364,56
0,15	1,8662	2,1833	2,8882	3,6877	4,5820	5,5711	6,6551	7,8341	15,156	24,862	51,437	133,27	505,18
0,20	2,0336	2,3968	3,2115	4,1445	5,1963	6,3670	7,6570	9,0662	17,906	29,741	62,407	163,74	627,06
0,25	2,1719	2,5737	3,4806	4,5262	5,7109	7,0352	8,4994	10,104	20,229	33,867	71,688	189,52	730,18
0,30	2,2834	2,7166	3,6987	4,8361	6,1296	7,5796	9,1864	10,950	22,129	37,242	79,282	210,61	814,56
0,35	2,3692	2,8267	3,8672	5,0761	6,4542	8,0020	9,7199	11,608	23,606	39,868	85,188	227,02	880,18
0,40	2,4301	2,9050	3,9871	5,2470	6,6856	8,3033	10,101	12,077	24,660	41,743	89,407	238,74	927,06
0,45	2,4665	2,9518	4,0589	5,3494	6,8243	8,4840	10,329	12,359	25,293	42,868	91,938	245,77	955,18
0,50	2,4786	2,9674	4,0828	5,3835	6,8705	8,5441	10,405	12,453	25,504	43,243	92,782	248,11	964,56

$$\varphi(\tau, \tau_0) \cong \frac{0.844+\tau_0}{1.421+\tau_0} + \left(1-2\,\frac{0.844+\tau_0}{1.421+\tau_0}\right)\frac{\tau}{\tau_0} \qquad (10)$$

may be used for $\varphi(\tau, \tau_0)$.

The accuracy of formula (10) improves as τ_0 increases. Expression (9) is, as has been mentioned earlier, a sufficiently accurate approximation of $\varphi_S(\tau, \tau_0)$ up to $\tau_0 = 1$. Assuming that a_1 and a_2 do not depend on τ, we will seek an expression for $\varphi_S(\tau, \tau_0)$ when $\tau_0 > 1$ in the form

$$\varphi_s(\tau, \tau_0) \cong \varphi_s(0, \tau_0)+a_1\tau+a_2\tau^2. \qquad (11)$$

Since

$$\frac{\partial[\varphi_s(\tau, \tau_0)]}{\partial\tau} = a_1 + 2a_2\tau, \qquad (12)$$

hence, by virtue of the symmetry of $\varphi_S(\tau, \tau_0)$ with respect to $\tau = \tau_0/2$,

$$a_1 = -a_2\tau_0.$$

Therefore,

$$\frac{\partial[\varphi_s(\tau, \tau_0)]}{\partial\tau} = -a_2(\tau_0-\tau). \qquad (13)$$

In order to determine a_2, we revert to Eq. (2). Integrating once in parts, we rewrite Eq. (2) as

$$1/2 + \varphi_s(0, \tau_0)\left[E_2(\tau) + E_2(\tau_0-\tau)\right] - \int_0^{\tau_0} \frac{\partial[\varphi_s(t, \tau_0)]}{\partial t}\,\mathrm{sign}(\tau-t)\,E_2(|\tau-t|)\,dt = 0. \qquad (14)$$

Inserting into Eq. (14) the value of the derivative from (13), we have

$$a_2 = \frac{\varphi_s(0, \tau_0)\left[E_2(\tau)+E_2(\tau_0-\tau)\right] - 1/2}{4/3-\tau_0\left[E_3(\tau)+E_3(\tau_0)\right] - 2\left[E_4(\tau)+E_4(\tau_0-\tau)\right]}.$$

TABLE 3. Relative Error in Determining the Radiant Flux according to Formulas (18)-(21) for $\varepsilon_1 = \varepsilon_2 = 1$

τ_0	$\Delta^{(1)}$, %	$\Delta^{(2)}$, %	$\Delta^{(3)}$, %	$\Delta^{(4)}$, %	$\Delta^{(1')}$, %
0,01	−0,10	−0,30	0	−1,01	−0,20
0,10	−0,66	−1,53	0,76	−9,17	−1,20
0,20	−0,71	−2,47	1,88	−17,8	−1,77
0,40	0	−3,08	4,29	−34,1	−1,88
0,60	0,15	−3,45	6,30	−50,0	−1,65
0,80	0	−3,31	8,10	−65,2	−1,32
1,00	−0,18	−3,25	9,59	−80,8	−1,09
1,50	−0,22	−3,06	12,5	−119	−0,22
2,00	0	−2,56	14,6	−156	0,51
3,00	0	−1,98	17,2	−65,5	1,66
5,00	0	−1,44	19,7	−20,2	−1,44
10,0	0	−0,86	22,2	5,13	−0,86

The expression for $\varphi_S(\tau, \tau_0)$, after a_1 and a_2 have been inserted into Eq. (11), will be

$$\varphi_s(\tau, \tau_0) \cong \varphi_s(0, \tau_0) + \tau(\tau_0 - \tau)\frac{1/2 - \varphi_s(0, \tau_0)\left[E_2(\tau) + E_2(\tau_0 - \tau)\right]}{4/3 - \tau_0\left[E_3(\tau) + E_3(\tau_0 - \tau)\right] - 2\left[E_4(\tau) + E_4(\tau_0 - \tau)\right]}. \quad (15)$$

Here $\varphi_S(\tau, \tau_0)$ can be approximated by the following expression [1]:

$$\varphi_s(0, \tau_0) \cong 0.305 + 0.217\,\tau_0.$$

We now consider the expression for the thermal flux through a plane layer of gray absorbing material. When the layer contains an active source of constant power S, then the expression for the thermal flux here becomes [1]

$$q(\tau) = \sigma(T_1^4 - T_2^4)\,Q(\tau_0)\left[1 + (1/\varepsilon_1 + 1/\varepsilon_2 - 2)\,Q(\tau_0)\right]^{-1}$$
$$+ S/k\left\{\tau - \left[\tau_0/2 - \tau_0(1/\varepsilon_2 - 1)\,Q(\tau_0)\right]\left[1 + (1/\varepsilon_1 + 1/\varepsilon_2 - 2)\,Q(\tau_0)\right]^{-1}\right\}. \quad (16)$$

Here function $Q(\tau_0)$ is defined according to (3).

We will derive for this case an approximate expression which is very accurate over the entire range of τ_0. We introduce the function

$$f(\tau_0) = \left[1 - Q(\tau_0)\right]/Q(\tau_0)$$

and we write Eq. (16) in terms of $f(\tau_0)$. Omitting here all intermediate steps, we write the final expression

$$q(\tau) = \frac{\sigma(T_1^4 - T_2^4)}{1/\varepsilon_1 + 1/\varepsilon_2 - 1 + f(\tau_0)} + \frac{S}{k}\left\{\tau - \frac{\tau_0/2\left[1 + f(\tau_0)\right] - \tau_0(1/\varepsilon_2 - 1)}{1/\varepsilon_1 + 1/\varepsilon_2 - 1 + f(\tau_0)}\right\}, \quad (17)$$

$$f(\tau_0) = 0.850\,\tau_0 \qquad (0 \leqslant \tau_0 < 0.5),$$
$$f(\tau_0) = 0.757\,\tau_0 + 0.047 \qquad (0.5 \leqslant \tau_0 \leqslant 3.0),$$
$$f(\tau_0) = 0.750\,\tau_0 + 0.066 \qquad (\tau_0 > 3.0).$$

The largest error in determining the thermal flux according to formula (17) is less than 1%.

In the special case where S = 0, Eq. (17) simplifies to

$$q^{(1)} = \sigma(T_1^4 - T_2^4)\left[1/\varepsilon_1 + 1/\varepsilon_2 - 1 + f(\tau_0)\right]^{-1}. \quad (18)$$

The following approximate formulas for the radiant flux through a layer with a zero source S = 0 have been published in the technical literature:

$$q^{(2)} = \sigma(T_1^4 - T_2^4)\left[1/\varepsilon_1 + 1/\varepsilon_2 - 1 + 3/4\,\tau_0\right]^{-1}, \quad (19)$$

$$q^{(3)} = \sigma(T_1^4 - T_2^4)\left[1/\varepsilon_1 + 1/\varepsilon_2 - 1 + \tau_0\right]^{-1}, \quad (20)$$

$$\begin{cases} q^{(4)} = \sigma(T_1^4 - T_2^4)\left[1/\varepsilon_1 + 1/\varepsilon_2 - 1\right]^{-1} & (\tau_0 \leqslant 2), \\ q^{(4)} = \sigma(T_1^4 - T_2^4)\left[1/\varepsilon_1 + 1/\varepsilon_2 - 3 + \tau_0\right]^{-1} & (\tau_0 > 2). \end{cases} \quad (21)$$

Radiant fluxes calculated by formulas (18)-(21) are compared in Table 3 with the values according to exact solution.

This comparison is based on the difference

$$\Delta^{(i)} = 100\,[q_T - q^{(i)}]/q_T \quad (i=1,\ 2,\ 3,\ 4)$$

for $\varepsilon_1 = \varepsilon_2 = 1$. Here q_T has been determined from Eq. (16) with S = 0.

According to Table 3, formula (18) yields the smallest error and formula (21) yields the largest error in the calculation of the radiant flux.

In the last column of Table 3 we compare the solution for the radiant flux according to formula (18) with the exact solution, but $f(\tau_0)$ is broken down into two ranges of optical thickness τ_0:

$$f(\tau_0) = 0.79\,\tau_0 \quad (0 \leqslant \tau_0 \leqslant 3.0),$$
$$f(\tau_0) = 0.75\,\tau_0 \quad (\tau_0 > 3.0).$$

The largest error in this case does not exceed 2%.

NOTATION

$E_n(z) = \int_1^\infty z^{-n} \exp(-z\mu)\,d\mu$ is the exponential integral of the n-th order;

k is the absorption coefficient, m^{-1};

L is the geometrical thickness, m;

S is the heat source, W/m^3;

T is the absolute temperature, °K;

t is the interpolation variable;

y is the geometrical distance, m;

ε is the emissivity;

σ is the Stefan–Boltzmann constant, $W/m^2\,(°K)^4$;

$\tau = \int_0^y k\,dy$ is the optical distance;

$\tau_0 = \int_0^L k\,dy$ is the optical thickness.

Subscripts

1 and 2 denote the upper surface and the lower surface.

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